Why is $e^{\pi\sqrt{163}}$ almost an integer?

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Ramanujan's constant

- $e^{\pi\sqrt{163}}$ is transcendental, but...
- Approximately

$$e^{\pi\sqrt{163}} \approx 262,537,412,640,768,743.999999999993...$$

It's decimal part is 7×10^{-13} away from the nearest integer.

- **Why is this?** This can be explained by
  modular forms + number theory.

Definitions: $\mathrm{SL}_2(\mathbb{Z})$

▶ Let $\mathbb{H}$ denote the complex upper half-plane. So

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{im}\,(z) > 0\}.$$

▶ Define the **special linear group** $\mathrm{SL}_2(\mathbb{Z})$ as

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_{2 \times 2}(\mathbb{Z}) \mid ad - bc = 1 \right\}.$$

▶ For $\gamma := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, let γ act on $\mathbb{H}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az + b}{cz + d}.$$

▶ One can check that $\gamma \cdot z \in \mathbb{H}$, and that this is indeed a **group action**.

Examples

For $z \in \mathbb{H}$;

- $T := \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot z = z + 1$; translation.

- $S := \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot z = -\frac{1}{z}$; double inversion.

- Fact: T and S generate $\mathrm{SL}_2(\mathbb{Z})$.

Definitions: modular forms

- A **modular form of weight** k is a function $f : \mathbb{H} \to C$ such that

  - $f(z)$ is analytic on $\mathbb{H}$;
  - $f(z)$ is "somewhat" $\mathrm{SL}_2(\mathbb{Z})$-invariant:

    $$\forall \gamma := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \forall z \in \mathbb{H},$$

    $$f(\gamma \cdot z) = (cz + d)^k f(z);$$

  - $f(z)$ is bounded as $\mathrm{im}(z) \to \infty$.

Fourier expansions

- Easy check: if $f(z)$ is a modular form, then one has for any $z \in \mathbb{H}$ that

$$f(T \cdot z) = (cz + d)^k f(z) = f(z),$$

i.e.,

$$f(z + 1) = f(z).$$

We deduce that $f(z)$ is **periodic**.

- Since $f(z)$ is periodic + analytic on $\mathbb{H}$, by complex analysis it may be written as

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \ q := e^{2\pi i z}.$$

This is called the **Fourier expansion** of $f(z)$.

Arithmetic in a modular form

- An **Eisenstein series of weight** k is the function $G_k : \mathbb{H} \to \mathbb{C}$ defined by

$$G_k(z) := \sum_{(c,d) \in \mathbb{Z}^2 \smallsetminus \{(0,0)\}} \frac{1}{(cz+d)^k}.$$

- With a bit of work, one can show that $G_k(z)$ converges absolutely, and is in fact a modular form.

- For even $k \geq 4$, $G_k(z)$ has the Fourier series expansion

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}$ and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$.

The j-invariant

▶ Define new modular forms

$$g_2(z) := 60\, G_4(z),\, g_3(z) := 160\, G_6(z).$$

▶ Next, define the *discriminant function*

$$\Delta(z) := g_2(z)^3 - 27 g_3(z)^2.$$

Δ is a weight 12 modular form which is nonzero on $\mathbb{H}$.

▶ Finally, define the **modular j-function** to be

$$j(z) := 1728 \frac{g_2^3(z)}{\Delta(z)}.$$

$j(z)$ is *not* a modular form: it is not analytic at ∞, since $j(z)$ has a q-expansion of the form

$$j(z) = \frac{1}{q} + \text{something analytic at } \infty.$$

The j-invariant

▶ A better look at the Fourier expansion is

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864229970q^3$$
$$+ 20245856256q^4 + 333202640600q^5 + \dots$$

▶ Miraculously, it can be shown that all q-coefficients of $j(z)$ are integers.

▶ Also interesting is that j is sometimes an algebraic integer:
$j(i) = 1728, j\left(\frac{1+\sqrt{-3}}{2}\right) = 0, j\left(\frac{1+\sqrt{-15}}{2}\right) =$
$-52515 - 85995\frac{1+\sqrt{5}}{2}$.

Wrapping up $e^{\pi\sqrt{163}}$

▶ So why is $e^{\pi\sqrt{163}}$ almost an integer?

▶ First, we note that **complex multiplication** tells us $j\left(\frac{1+\sqrt{-163}}{2}\right) \in \mathbb{Z}$.

▶ Recall $q := e^{2\pi i z}$; so setting $\tau := \frac{1+\sqrt{-163}}{2}$, we find that

$$\begin{aligned}
q(\tau) &= e^{\pi i(1+\sqrt{-163})} \\
&= e^{\pi i} \cdot e^{\pi\sqrt{163}} \\
&= (\cos(\pi) + i\sin(\pi)) \cdot e^{-\pi\sqrt{163}} \\
&= -e^{-\pi\sqrt{163}}.
\end{aligned}$$

Thus $q = -e^{-\pi\sqrt{163}} \approx -3.809 \times 10^{-18}$, which is pretty small.

Wrapping up $e^{\pi\sqrt{163}}$

▶ In particular, because $q \approx -3.809 \times 10^{-18}$ and $j(\tau)$ expands as

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864229970q^3 + \dots$$

Thus, the only dominating term is $\frac{1}{q} \approx -3.809 \times 10^{18}$, which is large in magnitude.

▶ Therefore, because $j(\tau)$ is also an integer, we may conclude that $\frac{1}{q} \approx j(\tau) - 744$ is also an integer.

▶ Thus so is $e^{\pi\sqrt{163}} = |q^{-1}|$.